1 Wave equation in a conductive medium

Maxwell's equations:

$$\begin{cases} \bar{\nabla}\cdot\bar{D} = \rho \\ \bar{\nabla}\cdot\bar{B} = 0 \\ \bar{\nabla}\times\bar{E} = -\dfrac{\partial\bar{B}}{\partial t} & \Big|\ \bar{\nabla}\times \\ \nabla\times\bar{H} = \bar{j} + \dfrac{\partial\bar{D}}{\partial t} & \Big|\ \dfrac{\partial}{\partial t} \end{cases}$$

Constitutive relations

$$\begin{cases} \bar{D} = \varepsilon_0\hat{\varepsilon}\bar{E} \\ \bar{B} = \mu_0\hat{\mu}\bar{H} \end{cases}$$

Ohm's law:

$$\bar{j} = \hat{\sigma}\bar{E}$$

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} = \mu_0\hat{\mu}\left(-\frac{\partial\bar{j}}{\partial t} - \frac{\partial^2\bar{D}}{\partial t^2}\right) = -\mu_0\hat{\mu}\hat{\sigma}\frac{\partial\bar{E}}{\partial t} - \mu_0\varepsilon_0\hat{\mu}\hat{\varepsilon}\frac{\partial^2\bar{E}}{\partial t^2}$$

Let's consider only plane waves:

$\bar{E} = \bar{E}_0 \exp(i(\bar{k}\bar{r} - \omega t))$ and transversal $(\bar{k}\cdot\bar{E}) = 0$

$$\Rightarrow k^2\bar{E} = \omega^2\mu_0\varepsilon_0\hat{\mu}\hat{\varepsilon}\bar{E} + i\omega\mu_0\hat{\mu}\hat{\sigma}\bar{E} = \quad \left(\mu_0\varepsilon_0 = \frac{1}{c^2}\right.$$

$$= \frac{\omega^2}{c^2}\left(\hat{\mu}\hat{\varepsilon} + i\,\frac{\hat{\mu}\hat{\sigma}}{\varepsilon_0\omega}\right)\bar{E} = k_0^2\,\hat{\tilde{\varepsilon}}\,\bar{E}$$

where $k_0 = \dfrac{\omega}{c}$ – wavevector in vacuum

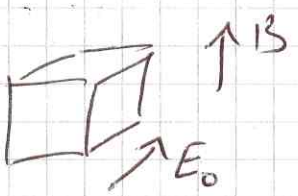
$$\hat{\tilde{\varepsilon}} = \hat{\tilde{n}}^2 = \hat{\mu}\hat{\varepsilon} + i\,\frac{\hat{\mu}\hat{\sigma}}{\varepsilon_0\omega}$$

complex dielectric permittivity — complex refractive index

$$\underline{\bar{k} = \hat{\tilde{n}}\,\bar{k}_0}$$

2. Dynamic tensor of magnetoconductivity

Crossed constant magnetic $\overline{B} = (0\ 0\ B)$ and periodic electric $\overline{E} = \overline{E}_0 \exp(-i\omega t)$ fields.



The dynamics of electron in a crystal subject to these fields is described by the Drude-Lorentz equation:

charge of the charge carriers

$$\left(\frac{d}{dt} + \frac{1}{\tau}\right)\overline{p} = q\left(\overline{E} + [\overline{v}\times\overline{B}]\right)$$

2nd Newton's law

characteristic relaxation time (mean time between scattering events)

Free electron model: $\overline{p} = m^*\overline{v}$

effective mass of the charge carriers.

Ansatz: $\overline{v} = \overline{v}_0 \exp(-i\omega t)$.

$$\frac{n\tau q}{m^*}\cdot\Big|\quad \frac{m^*}{\tau}(1 - i\omega\tau)\overline{v} = q\vec{E} + q\begin{pmatrix} v_y B \\ -v_x B \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 - i\omega\tau & -\omega_c\tau & 0 \\ \omega_c\tau & 1 - i\omega\tau & 0 \\ 0 & 0 & 1 - i\omega\tau \end{pmatrix}\overline{j} = \sigma_0\vec{E}$$

where $\omega_c = \frac{qB}{m^*}$ and $\sigma_0 = \frac{nq^2\tau}{m^*}$

Inverting this matrix, we get

$$\vec{j} = \sigma_0 \begin{pmatrix} \frac{1-i\omega\tau}{(1-i\omega\tau)^2+\omega_c^2\tau^2} & \frac{\omega_c\tau}{(1-i\omega\tau)^2+\omega_c^2\tau^2} & 0 \\ \frac{-\omega_c\tau}{(1-i\omega\tau)^2+\omega_c^2\tau^2} & \frac{1-i\omega\tau}{(1-i\omega\tau)^2+\omega_c^2\tau^2} & 0 \\ 0 & 0 & \frac{1}{1-i\omega\tau} \end{pmatrix} \vec{E} = \hat{\sigma}\vec{E}$$

So the conductivity tensor has the following structure:

$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ -\sigma_{xy} & \sigma_{xx} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

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3. Electromagnetic waves with circular polarization in a crystal subjected to external magnetic field

Faraday configuration: $\bar{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$, $\bar{B} \parallel Oz$

$$\bar{E}_{\pm} = \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \exp\left[i(kz - \omega t)\right], \quad \bar{k} \parallel Oz, \text{ polarized in } Oxy$$

$-$ correspond to left-handed polarization

$+$ to right-handed.

For simplicity, $\hat{\varepsilon} = \varepsilon$ & $\hat{\mu} = \mu$ (scalars), then

$$\hat{\varepsilon} = \mu\varepsilon + i\frac{\mu}{\varepsilon_0\omega}\hat{\sigma} = \begin{pmatrix} \mu\varepsilon + i\frac{\mu}{\varepsilon_0\omega}\sigma_{xx} & i\frac{\mu}{\varepsilon_0\omega}\sigma_{xy} & 0 \\ -i\frac{\mu}{\varepsilon_0\omega}\sigma_{xy} & \mu\varepsilon + i\frac{\mu}{\varepsilon_0\omega}\sigma_{xx} & 0 \\ 0 & 0 & \mu\varepsilon + i\frac{\mu}{\varepsilon_0\omega}\sigma_{zz} \end{pmatrix} =$$

$$= \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ -\varepsilon_{xy} & \varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}.$$ Inherits the symmetry of $\hat{\sigma}$

$$k^2 \bar{E}_{\pm} = \frac{\omega^2}{c^2}\hat{\varepsilon}\bar{E}_{\pm} \Rightarrow$$

$$\Rightarrow k^2 \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} = \frac{\omega^2}{c^2} \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ -\varepsilon_{xy} & \varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{k^2 = \frac{\omega^2}{c^2}(\varepsilon_{xx} \pm i\varepsilon_{xy})}$$

$$\boxed{\tilde{\varepsilon}_{\pm} = n_{\pm}^2 = \varepsilon_{xx} \pm i\varepsilon_{xy}}$$

Given the exact formulas for ε_{xx} ε_{xy}, σ_{xy} & σ_{xx}:

$$n_\pm^2 = \mu\varepsilon + \frac{i\mu\sigma_0}{\varepsilon_0\omega}\,\frac{(1-i\omega\tau)\pm i\omega_c\tau}{(1-i\omega\tau)^2+\omega_c^2\tau^2} =$$

$$= \mu\varepsilon + \frac{i\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1}{(1-i\omega\tau)\mp i\omega_c\tau} = \mu\varepsilon + \frac{i\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1}{1-i(\omega\pm\omega_c)\tau}$$

$$= \mu\varepsilon + \frac{i\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1+i(\omega\pm\omega_c)\tau}{1+(\omega\pm\omega_c)^2\tau^2} =$$

$$= \mu\varepsilon - \underbrace{\frac{\mu\sigma_0}{\varepsilon_0\omega}\,\frac{(\omega\pm\omega_c)\tau}{1+(\omega\pm\omega_c)^2\tau^2}}_{2\delta_\pm} + i\,\underbrace{\frac{\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1}{1+(\omega\pm\omega_c)^2\tau^2}}_{2\beta_\pm}$$

Let's assume $\delta, \beta \ll \mu\varepsilon$ (always true for X-rays)

$$\sqrt{x+\varepsilon} = \sqrt{x}\sqrt{1+\varepsilon/x} \approx \sqrt{x}\left(1+\frac{\varepsilon}{2x}\right) = \sqrt{x} + \frac{\varepsilon}{2\sqrt{x}}$$

Then $\boxed{n_\pm \simeq \sqrt{\mu\varepsilon} - \frac{\delta_\pm}{\sqrt{\mu\varepsilon}} + i\,\frac{\beta_\pm}{\sqrt{\mu\varepsilon}}}$

$$\bar{E}_\pm = \begin{pmatrix}1\\ \pm i\\ 0\end{pmatrix} \exp\left\{i\left[\underbrace{\left(\sqrt{\mu\varepsilon} - \frac{\delta_\pm}{\sqrt{\mu\varepsilon}}\right)\frac{\omega}{c}}_{k_{z\pm}} z - \omega t\right]\right\} \exp\left(-\underbrace{\frac{\beta_\pm \omega}{\sqrt{\mu\varepsilon}\, c}}_{\alpha_\pm} z\right)$$

$\alpha_\pm$ – attenuation coefficient

$$k_{z\pm} = \underbrace{\left(\frac{k_{z+}+k_{z-}}{2}\right)}_{\langle k_z\rangle} \pm \underbrace{\left(\frac{k_{z+}-k_{z-}}{2}\right)}_{\frac{\Delta k_z}{2}}$$

$$\boxed{\bar{E}_\pm = \begin{pmatrix}1\\ \pm i\\ 0\end{pmatrix} \exp\left(\pm i\frac{\Delta k_z}{2} z\right) \exp\left(-\alpha_\pm z\right) \exp\left[i\left(\langle k_z\rangle z - \omega t\right)\right]}$$

4. Faraday effect.

Let's assume $\alpha_\pm \approx 0$, then

$$\bar{E}_\pm = \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \exp\left(\pm i \frac{\Delta k_z}{2} z\right) \exp\left[i(\langle k_z\rangle z - \omega t)\right].$$

and

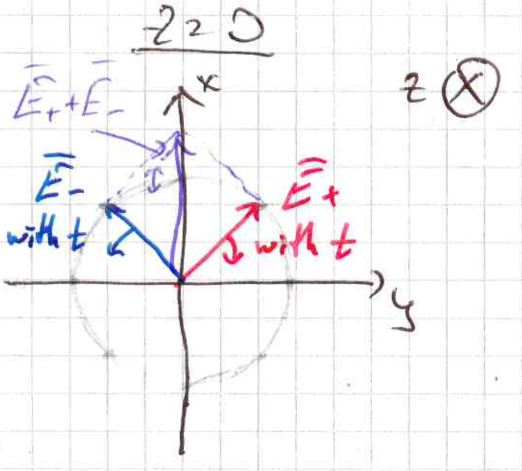
$$\bar{E}_+ + \bar{E}_- = \begin{pmatrix} \exp\left(i\frac{\Delta k_z}{2} z\right) + \exp\left(-i\frac{\Delta k_z}{2} z\right) \\ i\left[\exp\left(i\frac{\Delta k_z}{2} z\right) - \exp\left(-i\frac{\Delta k_z}{2} z\right)\right] \\ 0 \end{pmatrix} \exp\left[i(\langle k_z\rangle z - \omega t)\right]$$

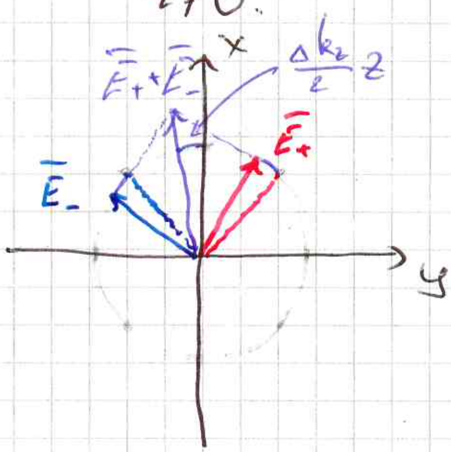
$$= \begin{pmatrix} 2\cos\left(\frac{\Delta k_z}{2} z\right) \\ -2\sin\left(\frac{\Delta k_z}{2} z\right) \\ 0 \end{pmatrix} \exp\left[i(\langle k_z\rangle z - \omega t)\right]$$

real vector!

$\Rightarrow$ linear polarization

For every fixed z it has linear polarization, but the polarization rotates along z!

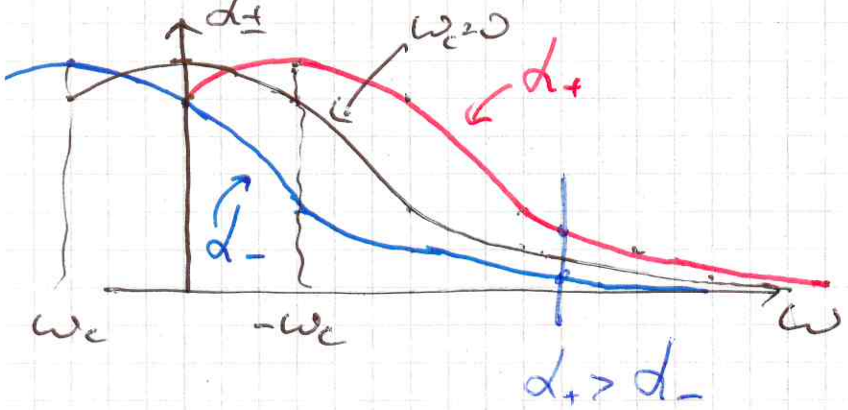




5. Magnetic circular dichroism

$$\alpha_\pm = \frac{\sqrt{\mu}\,\sigma_0}{2\sqrt{\varepsilon}\,\varepsilon_0}\,\frac{1}{1+(\omega\pm\omega_c)^2\tau^2}$$

For electrons $\omega_c < 0$, then



For electrons $\alpha_+ > \alpha_- \rightarrow$ right-handed polarization attenuated faster than left-handed

For holes $\omega_c > 0$ and $\alpha_+ < \alpha_- \rightarrow$ vice versa.

6. Dynamic tensor of magnetoconductivity near absorption edges

To take the absorption into account, let's modify the Drude-Lorentz equation:

$$\frac{d}{dt}\cdot\Big\{\; m\left(\frac{d}{dt}+\frac{1}{\tau}\right)\bar{v} = q\left(\vec{E}+[\bar{v}\times\bar{B}]\right) \underbrace{- m\omega_0^2\bar{r}}_{\text{restoring force for bound electrons.}}$$

$$\dot{\bar r} = \bar v$$

$$m\ddot{\bar v} + \frac{1}{\tau}\dot{\bar v} = q\dot{\vec E} + q[\dot{\bar v}\times\bar B] - m\omega_0^2\bar v$$

$$\bar E = \bar E_0\exp(-i\omega t),\quad \dot{\bar E} = -i\omega\bar E$$

$$\bar v = \bar v_0\exp(-i\omega t),\quad \dot{\bar v} = -i\omega\bar v,\quad \ddot{\bar v} = -\omega^2\bar v$$

$$-i\omega\frac{m}{\tau}\left(1-i\omega\tau\right)\vec v = -i\omega q\left(\bar E+[\bar v\times\bar B]\right) - m\omega_0^2\bar v$$

$$\cancel{-i\omega}\,\frac{m}{\tau}\left[1-i\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right)\right]\bar v = \cancel{-i\omega}\, q\left(\bar E+[\bar v\times\bar B]\right)$$

Can be obtained from normal magnetoconductivity tensor (section 2) by substitution

$$1-i\omega\tau \rightarrow 1-i\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right).$$

Everything else remains the same. Then,

$$\boxed{n_\pm^2 = \mu\varepsilon + \frac{i\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1-i\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right)\pm i\omega_c\tau}{\left[1-i\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right)\right]^2+\omega_c^2\tau^2}}$$

7. Absorption of circularly-polarized electromagnetic waves by bound electrons

$$n_\pm^2 = \mu\varepsilon + i\frac{\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1 - i\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right) \pm i\omega_c\tau}{\left[1 - i\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right)\right]^2 + \omega_c^2\tau^2} =$$

$$= \mu\varepsilon + i\frac{\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1}{1 - i\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right) \mp i\omega_c\tau} =$$

$$= \mu\varepsilon + i\frac{\mu\sigma_0}{\varepsilon_0\omega}\,\frac{1 + i\left(\omega\tau\left(1-\frac{\omega_0^2}{\omega^2}\right) \pm \omega_c\tau\right)}{1 + \left(\omega\left(1-\frac{\omega_0^2}{\omega^2}\right) \pm \omega_c\right)^2\tau^2} =$$

$$= \mu\varepsilon - \underbrace{\frac{\mu\sigma_0}{\varepsilon_0\cancel{\omega}}\,\frac{(\omega^2-\omega_0^2 \pm \omega\omega_c)\tau}{\omega^2 + (\omega^2-\omega_0^2 \pm \omega\omega_c)^2\tau^2}}_{2\delta_\pm} +$$

$$+ i\underbrace{\frac{\mu\sigma_0}{\varepsilon_0\cancel{\omega}}\,\frac{\omega}{\omega^2 + (\omega^2-\omega_0^2 \pm \omega\omega_c)^2\tau^2}}_{2\beta_\pm}$$

$$n_\pm \approx \sqrt{\mu\varepsilon} - \frac{\delta_\pm}{\sqrt{\mu\varepsilon}} + i\frac{\beta_\pm}{\sqrt{\mu\varepsilon}} \Rightarrow$$

$$\Rightarrow \boxed{\alpha_\pm = \frac{\beta_\pm}{\sqrt{\mu\varepsilon}}\frac{\omega}{c} = \frac{\sqrt{\mu}\,\sigma_0}{\sqrt{\varepsilon}\,\varepsilon_0 c}\cdot\frac{\omega^2}{\omega^2 + (\omega^2-\omega_0^2 \pm \omega\omega_c)^2\tau^2}}$$

Let's find the absorption maximum position.

$$\frac{\partial\alpha_\pm}{\partial\omega} = 0 = \frac{2\omega\left[\omega^2 + (\omega^2-\omega_0^2 \pm \omega\omega_c)^2\tau^2\right]}{\left[\omega^2 + (\omega^2-\omega_0^2 \pm \omega\omega_c)^2\tau^2\right]^2} -$$

$$-\frac{\omega^2\left[2\omega+2(\omega^2-\omega_0^2\pm\omega\omega_c)(2\omega\pm\omega_c)\tau^2\right]}{\left[\omega^2+(\omega^2-\omega_0^2\pm\omega\omega_c)^2\tau^2\right]^2}=0$$

$$\cancel{2\omega^3}+2\omega\tau^2(\omega^2-\omega_0^2\pm\omega\omega_c)^2-\cancel{2\omega^3}-$$
$$-2\omega(\omega^2-\omega_0^2\pm\omega\omega_c)^2(2\omega\pm\omega_c)\tau^2=0.$$

$$\omega(\omega^2-\omega_0^2\pm\omega\omega_c)\left[\omega^2-\omega_0^2\cancel{\pm\omega\omega_c}-2\omega^2\cancel{\mp\omega\omega_c}\right]=0$$

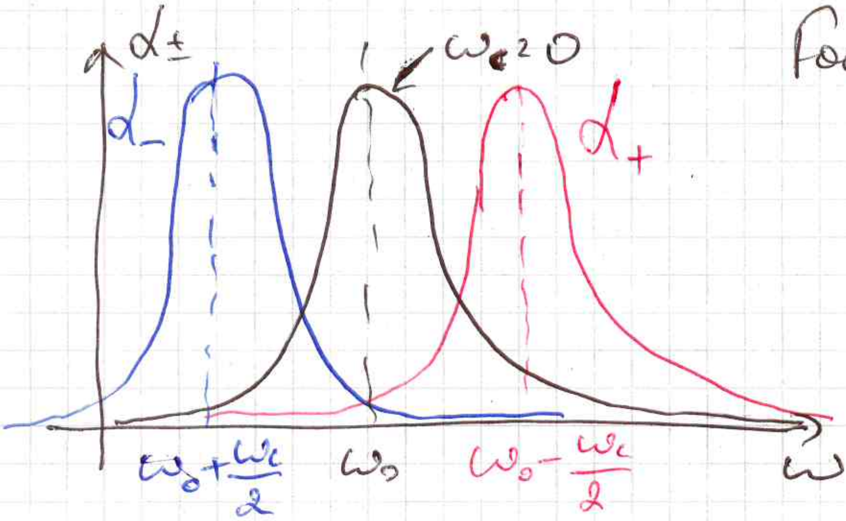
$\omega=0$ Trivial $\qquad\qquad$ $\omega^2=-\omega_0^2$ No real solution

$$\omega^2\pm\omega_c\omega-\omega_0^2=0$$

$$\omega_+^{1,2}=\frac{-\omega_c\pm\sqrt{4\omega_0^2+\omega_c^2}}{2}\qquad\omega_-^{1,2}=\frac{+\omega_c\pm\sqrt{4\omega_0^2+\omega_c^2}}{2}$$

We are interested only in positive roots

$$\omega_\pm=\frac{\sqrt{4\omega_0^2+\omega_c^2}\mp\omega_c}{2}\simeq\omega_0\mp\frac{\omega_c}{2}$$



For electrons $\omega_c<0$.

Right-handed polarization is absorbed at higher energies than left-handed.

This effect is called normal Zeeman effect

8. Quantum view on the Zeeman effect.

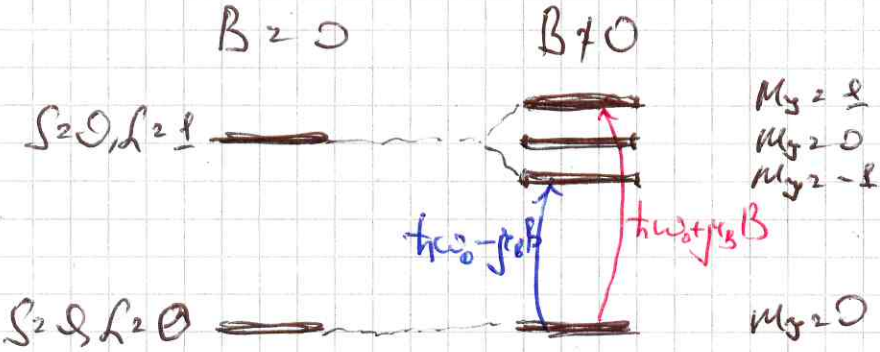
Let's look at the corresponding energies:

$\hbar\omega_{\pm} = \hbar\omega_0 \mp \frac{\hbar\omega_c}{2}$,

where $\omega_c = -\frac{eB}{m}$ and $\mu_B = \frac{\hbar e}{2m}$

Then, $\hbar\omega_{\pm} = \hbar\omega_0 \pm \mu_B B$.

From the quantum point of view, this transition happen between singlet ($S=0$) levels with $\Delta M_J = \pm 1$



Right-handed $\rightarrow$ $\Delta M_J = +1$

Left-handed $\rightarrow$ $\Delta M_J = -1$.

Circularly-polarized photons carry the spin angular momentum of ± 1.

Unfortunately, the classical theory does not explain the general anomalous Zeeman effect with $S \neq 0$ and $L \neq 0$. But the selection rules with $\Delta M_{\pm} = \pm 1$ for right (+) and left (−)

circular polarization holds true.

The probabilities of the corresponding transitions depend on the occupancies of the corresponding energy levels.

This makes the magnetic circular dichroism a very useful instrument to study magnetic materials:

1) It's element-selective (ω_0 depends on the atomic number).

2) It's orbital-selective (ω_0 depends on two sets of quantum numbers). Allows to study d- & f- orbitals contributing to magnetic properties.

3) It's sensitive to occupations of the corresponding orbitals.